Ejercicios Teoría Cuántica de Campos. Capítulo 81

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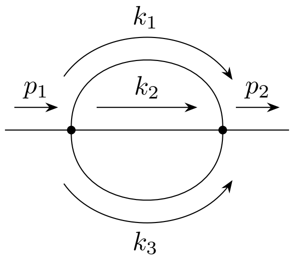
Sea el diagrama



1. Calcular el valor del diagrama en el espacio de momentos. Usando las reglas de Feynman

Las reglas de Feynman nos dicen que, para cada línea, debemos multiplicar por el propagador de Feynman. En éste diagrama tenemos 5 líneas, por lo que nos queda

$$\mathcal{A} = \frac{i}{p_1^2 - m^2 + i\varepsilon} \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - m^2 + i\varepsilon} \frac{i}{p_2^2 - m^2 + i\varepsilon} \tag{1}$$

Luego, cada vértice contribuye con un factor $(-i\lambda)\delta$, en nuestro caso tenemos dos vértices por lo que el resultado es

$$\begin{aligned}\mathcal{A} =& \frac{i}{p_1^2 - m^2 + i\varepsilon}(-i\lambda)(2\pi)^4\delta^{(4)}(p_1 - k_1 - k_2 - k_3)\frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - m^2 + i\varepsilon}\times \\ & (-i\lambda)(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3 - p_2)\frac{i}{p_2^2 - m^2 + i\varepsilon}\end{aligned} \tag{2}$$

La tercera regla es integrar todos los propagadores internos;

$$\begin{aligned}\mathcal{A} =& \int \frac{i}{p_1^2 - m^2 + i\varepsilon}(-i\lambda)(2\pi)^4\delta^{(4)}(p_1 - k_1 - k_2 - k_3)\frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - m^2 + i\varepsilon}\times \\ & (-i\lambda)(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3 - p_2)\frac{i}{p_2^2 - m^2 + i\varepsilon}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\end{aligned} \tag{3}$$

Finalmente debemos dividir entre el factor de simetría del diagrama (en este caso, 6)

$$\begin{aligned}\mathcal{A} =& \frac{1}{6}\int \frac{i}{p_1^2 - m^2 + i\varepsilon}(-i\lambda)(2\pi)^4\delta^{(4)}(p_1 - k_1 - k_2 - k_3)\frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - m^2 + i\varepsilon}\times \\ & (-i\lambda)(2\pi)^4\delta^{(4)}(k_1 + k_2 + k_3 - p_2)\frac{i}{p_2^2 - m^2 + i\varepsilon}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\end{aligned} \tag{4}$$

Simplificando esta expresión obtenemos

$$\mathcal{A} =\frac{-i\lambda^2}{6(2\pi)^4}\frac{1}{p_1^2-m^2+i\varepsilon}\frac{1}{p_2^2-m^2+i\varepsilon}\int\frac{1}{k_1^2-m^2+i\varepsilon}\frac{1}{k_2^2-m^2+i\varepsilon}\delta^{(4)}(k_1+k_2+k_3-p_2)\times$$
$$\frac{1}{k_3^2-m^2+i\varepsilon}\delta^{(4)}(p_1-k_1-k_2-k_3)\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3$$

E integrando k_3

$$\begin{aligned}\mathcal{A} =&\frac{-i\lambda^2}{6(2\pi)^4}\delta^{(4)}(p_1-p_2)\frac{1}{p_1^2-m^2+i\varepsilon}\frac{1}{p_2^2-m^2+i\varepsilon}\times\\ &\int\frac{1}{k_1^2-m^2+i\varepsilon}\frac{1}{k_2^2-m^2+i\varepsilon}\frac{1}{(p_1-k_1-k_2)^2-m^2+i\varepsilon}\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\end{aligned} \tag{5}$$

2. Calcular el valor del diagrama en el espacio de momentos. Usando la transformada de Fourier

Partiendo del valor del diagrama en espacio de posiciones:

$$\mathcal{A} = \frac{1}{6}(-i\lambda)^2\int\Delta_F(x-z_1)\Delta_F^3(z_1-z_2)\Delta_F(z_2-y)\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2 \tag{6}$$

Y haciendo la transformada de Fourier, teniendo en cuenta que en el diagrama p_1 sale de x y p_2 entra hacia y.

$$\begin{aligned}\mathcal{A} &= \frac{1}{6}(-i\lambda)^2\int e^{ip_1x}e^{-ip_2y}\Delta_F(x-z_1)\Delta_F^3(z_1-z_2)\Delta_F(z_2-y)\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4x\,\mathrm{d}^4y\\
&= \frac{(-i\lambda)^2}{6}\int e^{ip_1x}e^{-ip_2y}e^{-ik_1(x-z_1)}e^{-ik_2(z_1-z_2)}e^{-ik_3(z_1-z_2)}e^{-ik_4(z_1-z_2)}e^{-ik_5(z_2-y)}\times\\
&\quad\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\
&\quad\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4x\,\mathrm{d}^4y\\
&= \frac{(-i\lambda)^2}{6}\int e^{i(p_1-k_1)x}e^{i(-p_2+k_5)y}e^{i(k_1-k_2-k_3-k_4)z_1}e^{i(k_2+k_3+k_4-k_5)z_2}\times\\
&\quad\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\
&\quad\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\end{aligned}$$

Integrando respecto a x, y, z_1 y z_2 obtenemos varias deltas de Dirac

$$\begin{aligned}\mathcal{A} = \frac{(-i\lambda)^2}{6}&\int\delta^{(4)}(p_1-k_1)\delta^{(4)}(p_2+k_5)(2\pi)^4\delta^{(4)}(k_1-k_2-k_3-k_4)(2\pi)^4\delta^{(4)}(k_2+k_3+k_4-k_5)\times\\
&\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\
&\mathrm{d}^4k_1\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\,\mathrm{d}^4k_5\end{aligned}$$

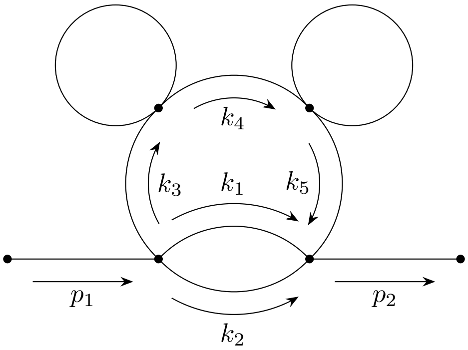
Finalmente podemos integrar respecto de k_1 y k_5

$$\begin{aligned}\mathcal{A} = \frac{(-i\lambda)^2}{6}&\int(2\pi)^4\delta^{(4)}(p_1-k_2-k_3-k_4)(2\pi)^4\delta^{(4)}(k_2+k_3+k_4-p_2)\times\\
&\frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\times\\
&\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\end{aligned}$$

Este expresión contiene exactamente los términos que aparecen debido a las reglas de Feynman (ecuación (4))

3. Bonus

Vamos a repetir el mismo ejercicio con el siguiente diagrama:



Por cada línea, multiplicamos por el propagador de Feynman;

$$\begin{aligned}\mathcal{A} =&\frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\ &\frac{i}{k_6^2-m^2+i\varepsilon}\frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\end{aligned}$$

Por cada vértice, multiplicamos por $(-i\lambda)(2\pi)^4\delta$

$$\begin{aligned}\mathcal{A} =&(-i\lambda)^4(2\pi)^{16}\delta^{(4)}(p_1-k_1-k_2-k_3)\delta^{(4)}(k_1+k_2+k_5-p_2)\delta^{(4)}(k_3-k_4)\delta^{(4)}(k_4-k_5)\times\\ &\frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\ &\frac{i}{k_6^2-m^2+i\varepsilon}\frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\end{aligned}$$

Integramos todos los cuadrimomentos internos;

$$\begin{aligned}\mathcal{A} =&\int(-i\lambda)^4(2\pi)^{16}\delta^{(4)}(p_1-k_1-k_2-k_3)\delta^{(4)}(k_1+k_2+k_5-p_2)\delta^{(4)}(k_3-k_4)\delta^{(4)}(k_4-k_5)\times\\ &\frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\ &\frac{i}{k_6^2-m^2+i\varepsilon}\frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\frac{\mathrm{d}^4k_6}{(2\pi)^4}\frac{\mathrm{d}^4k_7}{(2\pi)^4}\end{aligned}$$

Y finalmente dividimos entre el factor de simetría (en este caso, 8)

$$\begin{aligned}\mathcal{A} =&\frac{1}{8}\int(-i\lambda)^4(2\pi)^{16}\delta^{(4)}(p_1-k_1-k_2-k_3)\delta^{(4)}(k_1+k_2+k_5-p_2)\delta^{(4)}(k_3-k_4)\delta^{(4)}(k_4-k_5)\times\\ &\frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\times\\ &\frac{i}{k_6^2-m^2+i\varepsilon}\frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\frac{\mathrm{d}^4k_6}{(2\pi)^4}\frac{\mathrm{d}^4k_7}{(2\pi)^4}\end{aligned}$$

Simplificando las expresiones el resultado final queda

$$\begin{aligned}\mathcal{A} =&\frac{i\lambda^4}{8(2\pi)^{12}}\delta^{(4)}(p_1-p_2)\frac{1}{p_1^2-m^2+i\varepsilon}\frac{1}{p_2^2-m^2+i\varepsilon}\times\\ &\int\frac{1}{(p_1-k_1-k_2)^2-m^2+i\varepsilon}\frac{1}{k_1^2-m^2+i\varepsilon}\left(\frac{1}{k_2^2-m^2+i\varepsilon}\right)^3\frac{1}{k_3^2-m^2+i\varepsilon}\frac{1}{k_4^2-m^2+i\varepsilon}\times\\ &\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3\,\mathrm{d}^4k_4\end{aligned}$$

Si queremos hacerlo mediante la transformada de Fourier, empezamos con el diagrama en el espacio de posiciones

$$\begin{aligned}\mathcal{A} =&\frac{1}{8}(-i\lambda)^4\int\Delta_F(x-z_1)\Delta_F^2(z_1-z_2)\Delta_F(z_1-z_3)\Delta_F(z_3-z_3)\Delta_F(z_3-z_4)\Delta_F(z_4-z_4)\\ &\Delta_F(z_4-z_2)\Delta_F(z_2-y)\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4z_3\,\mathrm{d}^4z_4\end{aligned}$$

Haciendo la transformada de Fourier

$$\begin{aligned}\mathcal{A} = \frac{1}{8}(-i\lambda)^4 \int & e^{ip_1x}e^{-ip_2y}\Delta_F(x-z_1)\Delta_F^2(z_1-z_2)\Delta_F(z_1-z_3)\Delta_F(z_3-z_3)\Delta_F(z_3-z_4) \\ & \Delta_F(z_4-z_4)\Delta_F(z_4-z_2)\Delta_F(z_2-y)\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4z_3\,\mathrm{d}^4z_4\,\mathrm{d}^4x\,\mathrm{d}^4y \\ = \frac{1}{8}(-i\lambda)^4 \int & e^{ip_1x}e^{-ip_2y}e^{-ik_1(x-z_1)}e^{-ik_2(z_1-z_2)}e^{-ik_3(z_1-z_2)}e^{-ik_4(z_1-z_3)}e^{-ik_5(z_3-z_3)}e^{-ik_6(z_3-z_4)} \\ & e^{-ik_7(z_4-z_4)}e^{-ik_8(z_4-z_2)}e^{-ik_9(z_2-y)}\frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon} \\ & \frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\frac{i}{k_6^2-m^2+i\varepsilon}\frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{k_8^2-m^2+i\varepsilon}\frac{i}{k_9^2-m^2+i\varepsilon} \\ & \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\frac{\mathrm{d}^4k_6}{(2\pi)^4}\frac{\mathrm{d}^4k_7}{(2\pi)^4}\frac{\mathrm{d}^4k_8}{(2\pi)^4}\frac{\mathrm{d}^4k_9}{(2\pi)^4}\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4z_3\,\mathrm{d}^4z_4\,\mathrm{d}^4x\,\mathrm{d}^4y \\ = \frac{1}{8}(-i\lambda)^4 \int & e^{i(p_1-k_1)x}e^{i(-p_2+k_9)y}e^{i(k_1-k_2-k_3-k_4)z_1}e^{i(k_2+k_3+k_8-k_9)z_2}e^{i(k_4-k_6)z_3}e^{i(k_6-k_7)z_4} \\ & \frac{i}{k_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\frac{i}{k_6^2-m^2+i\varepsilon} \\ & \frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{k_8^2-m^2+i\varepsilon}\frac{i}{k_9^2-m^2+i\varepsilon}\,\mathrm{d}^4z_1\,\mathrm{d}^4z_2\,\mathrm{d}^4z_3\,\mathrm{d}^4z_4\,\mathrm{d}^4x\,\mathrm{d}^4y \\ & \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\frac{\mathrm{d}^4k_6}{(2\pi)^4}\frac{\mathrm{d}^4k_7}{(2\pi)^4}\frac{\mathrm{d}^4k_8}{(2\pi)^4}\frac{\mathrm{d}^4k_9}{(2\pi)^4}\end{aligned}$$

Integrando x, y y z_i obtenemos las deltas de Dirac

$$\begin{aligned}\mathcal{A} = & \frac{1}{8}(-i\lambda)^4 \int (2\pi)^{16}\delta^{(4)}(p_1-k_2-k_3-k_4)\delta^{(4)}(k_2+k_3+k_8-p_2)\delta^{(4)}(k_4-k_6)\delta^{(4)}(k_6-k_7) \\ & \frac{i}{p_1^2-m^2+i\varepsilon}\frac{i}{k_2^2-m^2+i\varepsilon}\frac{i}{k_3^2-m^2+i\varepsilon}\frac{i}{k_4^2-m^2+i\varepsilon}\frac{i}{k_5^2-m^2+i\varepsilon}\frac{i}{k_6^2-m^2+i\varepsilon} \\ & \frac{i}{k_7^2-m^2+i\varepsilon}\frac{i}{k_8^2-m^2+i\varepsilon}\frac{i}{p_2^2-m^2+i\varepsilon}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{\mathrm{d}^4k_3}{(2\pi)^4}\frac{\mathrm{d}^4k_4}{(2\pi)^4}\frac{\mathrm{d}^4k_5}{(2\pi)^4}\frac{\mathrm{d}^4k_6}{(2\pi)^4}\frac{\mathrm{d}^4k_7}{(2\pi)^4}\frac{\mathrm{d}^4k_8}{(2\pi)^4}\end{aligned}$$

que es justo lo que obtenemos al aplicar las reglas de Feynman.